

New results on p -Bernoulli numbers

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Abstract

We realize that geometric polynomials and p -Bernoulli polynomials and numbers are closely related with an integral representation. Therefore, using geometric polynomials, we extend some properties of Bernoulli polynomials and numbers such as recurrence relations, telescopic formula and Raabe's formula to p -Bernoulli polynomials and numbers. In particular cases of these results, we establish some new results for Bernoulli polynomials and numbers. Moreover, we evaluate a Faulhaber-type summation in terms of p -Bernoulli polynomials.

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1 Introduction

The Bernoulli polynomials $B_n(x)$ are defined by exponential generating function

$$\sum_{n \geq 0} B_n(x) \frac{t^n}{n!} = \frac{te^{xt}}{e^t - 1}, \quad |t| < 2\pi. \quad (1)$$

In particular, the rational numbers $B_n = B_n(0)$ are called Bernoulli numbers and have an explicit formula [14]

$$B_n = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{(-1)^k k!}{k+1},$$

where $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is the Stirling number of second kind [14].

As it is well known, the Bernoulli numbers are considerable importance in different areas of mathematics such as number theory, combinatorics, special functions. Moreover, many generalizations and extensions of these numbers appear in the literature. One of the generalization of the Bernoulli numbers is

p -Bernoulli numbers, defined by a three-term recurrence relation [23]

$$B_{n+1,p} = pB_{n,p} - \frac{(p+1)^2}{p+2} B_{n,p+1}, \quad (2)$$

with the initial condition $B_{0,p} = 1$. These numbers also satisfy an explicit formula

$$B_{n,p} = \frac{p+1}{p!} \sum_{k=0}^n \left\{ \begin{matrix} n+p \\ k+p \end{matrix} \right\}_p \frac{(-1)^k (k+p)!}{k+p+1},$$

where $\left\{ \begin{matrix} n+p \\ k+p \end{matrix} \right\}_p$ is the p -Stirling number of second kind [6].

As a special case, setting $p = 0$ in the above equation gives $B_{n,0} = B_n$.

p -Bernoulli polynomials which is the polynomial extension of $B_{n,p}$, are defined by the following convolution

$$B_{n,p}(x) = \sum_{k=0}^n \binom{n}{k} x^{n-k} B_{k,p}. \quad (3)$$

The first few p -Bernoulli polynomials are

$$\begin{aligned} B_{0,p}(x) &= 1, \\ B_{1,p}(x) &= x - \frac{1}{p+2}, \\ B_{2,p}(x) &= x^2 - \frac{2x}{p+2} - \frac{p-1}{(p+2)(p+3)}. \end{aligned}$$

Moreover, these polynomials have integral representations

$$\int_b^a B_{n,p}(t) dt = \frac{B_{n+1,p}(a) - B_{n+1,p}(b)}{n+1}, \quad (4)$$

$$\int_0^1 B_{n,p}(t) dt = \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k} B_{k,p}, \quad (5)$$

a recurrence relation

$$B_{n,p}(x+1) - B_{n,p}(x) = \sum_{k=0}^{n-1} \binom{n}{k} B_{k,p}(x), \quad (6)$$

and a three-term recurrence relation

$$B_{n+1,p}(x) = (x+p) B_{n,p}(x) - \frac{(p+1)^2}{p+2} B_{n,p+1}(x). \quad (7)$$

In the special case of (3) when $x = 0$, we obtain $B_{n,p}(0) = B_{n,p}$.

Some other properties and applications of p -Bernoulli polynomials and numbers can be found in [23].

The main formula of this paper is [23, p. 361]

$$\frac{1}{p+1} \sum_{n \geq 0} B_{n,p} \frac{t^n}{n!} = \int_{-1}^0 \frac{(1+y)^p}{1-y(e^t-1)} dy, \text{ for } p \geq 0.$$

Using the generating function of geometric polynomials $w_n(y)$ (see Section 2 for details of $w_n(y)$), the above equation can be written as

$$\frac{1}{p+1} B_{n,p} = \int_{-1}^0 (1+y)^p w_n(y) dy \quad (8)$$

which is the generalization of Keller's identity [16]

$$\int_{-1}^0 w_n(y) dy = B_n.$$

Thus, using this integral representation and the properties of geometric polynomials, we generalize a recurrence relation of Bernoulli numbers to p -Bernoulli numbers and obtain an explicit formula for p -Bernoulli numbers. Moreover, extending the representation (8) to p -Bernoulli polynomials, we give the generalization of the telescopic formula and Raabe's formula of Bernoulli polynomials for p -Bernoulli polynomials. Thus, as special cases of these results, we get an explicit formula, a finite summation and a convolution identity for Bernoulli polynomials and numbers. Besides, we evaluate a Faulhaber-type summation in terms of p -Bernoulli polynomials.

First, we extend the well known recurrence relation of Bernoulli numbers

$$\sum_{k=0}^n \binom{n+1}{k} B_k = 0 \text{ for } n \geq 1,$$

to p -Bernoulli numbers in the following theorem.

Theorem 1 For $n \geq 1$ and $p \geq 0$,

$$\sum_{k=0}^n \binom{n+1}{k} B_{k,p} = -p B_{n,p}. \quad (9)$$

We note that using (5) and (6) in the above theorem gives us the following conclusions

$$B_{n,p}(1) = B_{n,p} - p B_{n-1,p},$$

and

$$\int_0^1 B_{n,p}(t) dt = \frac{-p B_{n,p}}{n+1}, \quad (10)$$

respectively. Also, these results are the generalization of the following well known properties of B_n

$$B_n(1) = B_n \text{ and } \int_0^1 B_n(t) dt = 0, \text{ for } n \geq 1.$$

The Bernoulli numbers are connected with some well known special numbers [7, 8, 18, 19, 20, 21]. Rahmani [23] also gave explicit formulas involving different kind of special numbers. Now, we obtain a new explicit formula for $B_{n,p}$, and hence B_n , in the following theorem.

Theorem 2 For $n \geq 1$ and $p \geq 0$,

$$B_{n,p} = (p+1) \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{(-1)^{k+n} k!}{(k+p)(k+p+1)}. \quad (11)$$

When $p = 0$ this becomes

$$B_n = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{(-1)^{k+n} (k-1)!}{k+1}. \quad (12)$$

In order to deal with some properties of p -Bernoulli polynomials, we need to extend the integral representation (8) to $B_{n,p}(x)$.

Proposition 3 Let n and p be the non-negative integers. Then we have

$$\frac{1}{p+1} B_{n,p}(x) = \int_{-1}^0 (1+y)^p w_n(x; y) dy, \quad (13)$$

where $w_n(x; y)$ (see Section 2) is two variable geometric polynomials.

One of the important properties of $B_n(x)$ is the telescopic formula

$$B_n(x+1) - B_n(x) = nx^{n-1}.$$

From this formula, Bernoulli gave a closed formula for Faulhaber's summation in terms of Bernoulli polynomials and numbers

$$\sum_{k=0}^m k^n = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1}. \quad (14)$$

Now, we want to give an extension of telescopic formula for p -Bernoulli polynomials.

Proposition 4 For any non-negative integer n and p ,

$$B_{n,p+1}(x+1) - B_{n,p+1}(x) = \frac{p+2}{p+1} (B_{n,p}(x+1) - x^n). \quad (15)$$

This telescopic formula for p -Bernoulli polynomials gives us the evaluation of finite summation of p -Bernoulli polynomials. In particular case $p = 0$, we arrive at a new finite summation involving Bernoulli polynomials.

Theorem 5 *For any non-negative integer n, m and p ,*

$$\sum_{k=0}^m B_{n,p}(k+1) = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1} + \frac{p+1}{p+2} (B_{n,p+1}(m+1) - B_{n,p+1}). \quad (16)$$

When $p = 0$ this becomes

$$\sum_{k=0}^m [B_n(k+1) + nk^n] = (m+1) B_n(m+1). \quad (17)$$

Another important identity for Bernoulli polynomials is the Raabe's formula

$$m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right) = B_n(mx).$$

Now, we want to extend the Raabe's formula to p -Bernoulli polynomials.

Theorem 6 *For $m \geq 1$ and $n, p \geq 0$,*

$$m^{n-1} \sum_{k=0}^{m-1} B_{n,p}\left(x + \frac{k}{m}\right) = (p+1) B_n(mx) - p \sum_{k=0}^n \binom{n}{k} \frac{m^k B_{n-k}(mx) B_{k,p}}{k+1}. \quad (18)$$

Using the generating function technique, Chu and Zhou [9] give several convolution identities for Bernoulli numbers. Two of them are the followings:

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} \frac{B_{k+1} B_{n-k}}{k+1} &= -B_n - B_{n+1}, \\ \sum_{k=0}^n \binom{n}{k} \frac{2^k B_{k+1} B_{n-k}}{k+1} &= \frac{-B_{n+1} - (2^{n-1} + 1) B_n}{2}. \end{aligned}$$

If we set $p = 1$ and $x = 0$ in (18) and use (2) and (8), we have a close formula for a generalization of the above equations in the following corollary.

Corollary 7 *For $m \geq 1$ and $n, p \geq 0$,*

$$\sum_{k=0}^n \binom{n}{k} \frac{m^k B_{k+1} B_{n-k}}{k+1} = \frac{-m B_n - B_{n+1}}{m} + m^{n-1} \sum_{k=0}^{m-1} \frac{k}{m} B_n\left(\frac{k}{m}\right).$$

Finally, we evaluate a Faulhaber-type summation in terms of p -Bernoulli polynomials and numbers which generalize the following finite summation [15, p. 18, Eq. 1]

$$\sum_{k=0}^n \frac{(-1)^k}{\binom{n}{k}} = \frac{n+1}{n+2} ((-1)^n + 1).$$

Theorem 8 For $n \geq 1$ and $p \geq 0$, we have

$$\sum_{k=0}^n \frac{k^p (-1)^k}{\binom{n}{k}} = \frac{n+1}{n+2} \left[(-1)^{n+p} B_{p,n+1}(-n) + B_{p,n+1} \right].$$

The summary by sections is as follows: Section 2 is the preliminary section where we give definitions and known results needed. In Section 3, we derive a recurrence relation for p -Bernoulli and a Raabe-type relation for geometric polynomials, which we need in the proofs of Theorem 6 and Theorem 8. In Section 4, we give the proofs of the results, mentioned above.

2 Preliminaries

Geometric polynomials are defined by the exponential generating function [22]

$$\frac{1}{1-y(e^t-1)} = \sum_{n=0}^{\infty} w_n(y) \frac{t^n}{n!}. \quad (19)$$

They have an explicit formula

$$w_n(y) = y \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (-1)^{n+k} k! (y+1)^{k-1}, \quad n > 0, \quad (20)$$

and a reflection formula

$$w_n(y) = (-1)^n \frac{y}{y+1} w_n(-y-1), \quad \text{for } n > 0, \quad (21)$$

Moreover, these polynomials are related to p -Bernoulli numbers with an integral representation

$$\int_{-1}^0 y^p w_n(y) dy = (-1)^{n+p+1} \frac{p+1}{p+2} B_{n-1,p+1}, \quad \text{for } n > 1, p \geq 0. \quad (22)$$

See [1, 2, 3, 4, 5, 13, 17] for other properties and applications of geometric polynomials.

Two variable geometric polynomials are defined by means of the following generating function [17]

$$\sum_{n=0}^{\infty} w_n(x; y) \frac{t^n}{n!} = \frac{e^{xt}}{1-y(e^t-1)}. \quad (23)$$

Moreover, they are related to $w_k(y)$ with a convolution

$$w_n(x; y) = \sum_{k=0}^n \binom{n}{k} w_k(y) x^{n-k}, \quad (24)$$

with a special case

$$w_n(0; y) = w_n(y). \quad (25)$$

3 Some other basic properties

In this section, in order to use in the proof of Theorem 6 and Theorem 8, we give a recurrence relation for p -Bernoulli polynomials and a Raabe-type formula for two variable geometric polynomials.

For the proof of Theorem 6, we first need the following proposition.

Proposition 9 *For $n \geq 1$ and $p \geq 0$, we have*

$$p^2 \sum_{k=1}^n \binom{n+1}{k+1} y^{n-k} B_{k,p} = (p+1) y^{n+1} + p(n+1) y^n - (p+1) B_{n+1,p-1} (1+y). \quad (26)$$

Proof. From (3), we have

$$\sum_{k=1}^n \binom{n}{k} y^{n-k} B_{k,p}(x) = B_{n,p}(x+y) - y^n. \quad (27)$$

Let integrate both sides of the above equation with respect to x from 0 to 1. Then, using (10), the left hand side of (27) becomes

$$\begin{aligned} \sum_{k=1}^n \binom{n}{k} y^{n-k} \int_0^1 B_{k,p}(x) dx &= -p \sum_{k=1}^n \binom{n}{k} \frac{y^{n-k} B_{k,p}}{k+1} \\ &= \frac{-p}{n+1} \sum_{k=1}^n \binom{n+1}{k+1} y^{n-k} B_{k,p}. \end{aligned} \quad (28)$$

On the other hand, using (4) and Proposition 4 in the right hand side of (27), we have

$$\begin{aligned} \int_0^1 [B_{n,p}(x+y) - y^n] dx &= \int_y^{y+1} B_{n,p}(t) dt - y^n \int_0^1 dx \\ &= \frac{B_{n+1,p}(y+1) - B_{n+1,p}(y)}{n+1} - y^n \\ &= \frac{p+1}{p(n+1)} [B_{n+1,p-1}(y+1) - y^{n+1}] - y^n. \end{aligned}$$

Combining the above equation with (28) gives the desired equation. ■

Now, we give the Raabe-type formula for two variable geometric polynomials in the following proposition. Later, we use it in the proof of Theorem 8.

Proposition 10 *For $m \geq 1$ and $n, p \geq 0$,*

$$m^{n-1} \sum_{k=0}^{m-1} w_{n-1} \left(x + \frac{k}{m}, y \right) = \frac{1}{ny} \sum_{k=1}^n \binom{n}{k} m^k B_{n-k}(mx) w_k(y). \quad (29)$$

Proof. Using (23) and the identity

$$\sum_{k=0}^{m-1} x^k = \frac{x^m - 1}{x - 1},$$

we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{m-1} w_n \left(x + \frac{k}{m}, y \right) &= \sum_{k=0}^{m-1} \sum_{n=0}^{\infty} w_n \left(x + \frac{k}{m}, y \right) \frac{t^n}{n!} \\ &= \frac{1}{1 - y(e^t - 1)} \sum_{k=0}^{m-1} e^{(x + \frac{k}{m})t} \\ &= \frac{e^{xt}}{1 - y(e^t - 1)} \frac{e^t - 1}{e^{t/m} - 1} \\ &= \frac{e^{xt}}{e^{t/m} - 1} \frac{1}{1 - y(e^t - 1)} \frac{(1 - (1 - y(e^t - 1)))}{y} \\ &= \frac{1}{y(t/m)} \left(\frac{(t/m)e^{xt}}{e^{t/m} - 1} \frac{1}{1 - y(e^t - 1)} - \frac{(t/m)e^{xt}}{e^{t/m} - 1} \right). \end{aligned}$$

From (1) and (19), the above equation can be written as

$$\frac{yn}{m} \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{m-1} w_{n-1} \left(x + \frac{k}{m}, y \right) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \left[\sum_{k=1}^n \binom{n}{k} \frac{B_{n-k}(mx) w_k(y)}{m^{n-k}} - \frac{B_n(mx)}{m^n} \right].$$

Finally, comparing the coefficients of $\frac{t^n}{n!}$ in the both sides of the above equation, we get (29). ■

4 Proofs

In this section, we give the proofs of all results mentioned in Section 1.

Proof of Theorem 1. Using (21) in the following equation [5, Proposition 15], we have

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} w_k(y) &= \frac{1+y}{y} w_n(y) \\ &= (-1)^n w_n(-y-1). \end{aligned}$$

Multiplying both sides of the above equation by $(1+y)^p$, integrating it with

respect to y from -1 to 0 and using (8) and (22), we achieve

$$\begin{aligned}
\frac{1}{p+1} \sum_{k=0}^n \binom{n}{k} B_{k,p} &= (-1)^n \int_{-1}^0 (1+y)^p w_n(-y-1) dy \\
&= (-1)^{n+p} \int_{-1}^0 x^p w_n(x) dx \\
&= -\frac{p+1}{p+2} B_{n-1,p+1}.
\end{aligned}$$

Finally, using (2) gives the desired equation. ■

Proof of Theorem 2. Multiplying both sides of (20) by $(1+y)^p$, integrating it with respect to y from -1 to 0 and using (8), we have

$$\begin{aligned}
\frac{1}{p+1} B_{n,p} &= \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (-1)^{n+k} k! \int_{-1}^0 y (y+1)^{p+k-1} dy \\
&= \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (-1)^{n+k+1} k! \int_0^1 (1-x) x^{p+k-1} dx.
\end{aligned}$$

Finally, using the well known relation of Beta function

$$B(x, y) = \int_0^1 (1-t)^{x-1} t^{y-1} dt = \frac{(x-1)! (y-1)!}{(x+y-1)!}, \quad (30)$$

where $x, y = 1, 2, 3, \dots$, we obtain (11). ■

Proof of Proposition 3. Using the equations (8) and (3) in (24), we have

$$\begin{aligned}
\int_{-1}^0 (1+y)^p w_n(x; y) dy &= \sum_{k=0}^n \binom{n}{k} x^{n-k} \int_{-1}^0 (1+y)^p w_k(y) dy \\
&= \sum_{k=0}^n \binom{n}{k} x^{n-k} B_{k,p} \\
&= \frac{1}{p+1} B_{n,p}.
\end{aligned}$$

■

Proof of Proposition 4. The two variable geometric polynomials have [17, Eq. 14]

$$w_n(x+1; y) - w_n(x; y) = \frac{1}{1+y} (w_n(x+1, y) - x^n).$$

Multiplying both sides of the above equation by $(1+y)^{p+1}$, integrating it with respect to y from -1 to 0 and using (8) yield (15). ■

Proof of Theorem 5. Replacing x with k in (15) and summing over k from 0 to m , we obtain

$$\begin{aligned} \frac{p+2}{p+1} \sum_{k=0}^m B_{n,p}(k+1) - \sum_{k=0}^m k^n &= \sum_{k=0}^m (B_{n,p+1}(k+1) - B_{n,p+1}(k)) \\ &= B_{n,p+1}(m+1) - B_{n,p+1}. \end{aligned} \quad (31)$$

If we use Bernoulli's well known identity for Faulhaber summation

$$\sum_{k=0}^m k^n = \frac{B_{n+1}(m+1) + B_{n+1}}{n+1},$$

in the second part of the left hand side of (31), we arrive at the first part of theorem.

For the second part of the theorem, if we use (2) and (7) for $p = 0$, (16) becomes

$$\begin{aligned} \sum_{k=0}^m B_n(k+1) &= (m+1) B_n(m+1) - n \frac{B_{n+1}(m+1) + B_{n+1}}{n+1} \\ &= (m+1) B_n(m+1) - n \sum_{k=0}^m k^n. \end{aligned}$$

Then, we have (17). ■

Proof of Theorem 6. Multiplying both sides of (29) by $(1+y)^{p+1}$ and using (21), we have

$$\begin{aligned} m^{n-1} \sum_{k=0}^{m-1} (1+y)^{p+1} w_{n-1} \left(x + \frac{k}{m}, y \right) \\ &= \frac{1}{n} \sum_{k=1}^n \binom{n}{k} m^k B_{n-k}(mx) \frac{(1+y)^{p+1}}{y} w_k(y) \\ &= \frac{1}{n} \sum_{k=1}^n \binom{n}{k} m^k B_{n-k}(mx) (1+y)^p (-1)^k w_k(-y-1) \\ &= m B_{n-1}(mx) (1+y)^{p+1} + \frac{1}{n} \sum_{k=2}^n \binom{n}{k} m^k B_{n-k}(mx) (-1)^k (1+y)^p w_k(-y-1). \end{aligned}$$

Integrating the above equation with respect to y from -1 to 0 and using (8)

and (22), we obtain

$$\begin{aligned}
& \frac{m^{n-1}}{p+2} \sum_{k=0}^{m-1} B_{n-1,p+1} \left(x + \frac{k}{m} \right) \\
&= \frac{mB_{n-1}(mx)}{p+2} + \frac{1}{n} \sum_{k=2}^n \binom{n}{k} m^k B_{n-k}(mx) (-1)^k \int_{-1}^0 (1+y)^p w_k(-y-1) dy \\
&= \frac{mB_{n-1}(mx)}{p+2} + \frac{1}{n} \sum_{k=2}^n \binom{n}{k} m^k B_{n-k}(mx) (-1)^{k+p} \int_{-1}^0 x^p w_k(x) dx \\
&= \frac{mB_{n-1}(mx)}{p+2} - \frac{p+1}{n(p+2)} \sum_{k=2}^n \binom{n}{k} m^k B_{n-k}(mx) B_{k-1,p+1} \\
&= mB_{n-1}(mx) - \frac{p+1}{n(p+2)} \sum_{k=1}^n \binom{n}{k} m^k B_{n-k}(mx) B_{k-1,p+1}.
\end{aligned}$$

Replacing $p+1$ with p and n with $n+1$, the above equation can be rewritten as

$$\begin{aligned}
m^{n-1} \sum_{k=0}^{m-1} B_{n,p} \left(x + \frac{k}{m} \right) &= (p+1) B_n(mx) - \frac{p}{n+1} \sum_{k=1}^{n+1} \binom{n+1}{k} m^{k-1} B_{n+1-k}(mx) B_{k-1,p} \\
&= (p+1) B_n(mx) - \frac{p}{n+1} \sum_{k=0}^n \binom{n+1}{k+1} m^k B_{n-k}(mx) B_{k,p} \\
&= (p+1) B_n(mx) - p \sum_{k=0}^n \binom{n}{k} \frac{m^k B_{n-k}(mx) B_{k,p}}{k+1}.
\end{aligned}$$

■

Proof of Theorem 8. We have an arithmetic-geometric progression [11, 24]

$$\sum_{k=0}^n k^p y^k = \frac{A_p(y)}{(1-y)^{p+1}} - y^{n+1} \sum_{k=0}^p \binom{p}{k} (n+1)^{p-k} \frac{A_k(y)}{(1-y)^{k+1}}, \quad (32)$$

where $A_n(y)$ is the Eulerian polynomial of degree n [10]. These polynomials are closely related to the geometric polynomials with relation [1, Eq. 3.18]

$$A_n(y) = (1-y)^n w_n(y).$$

If we multiply both side of (32) and use this relation, (32) can be rewritten as

$$\sum_{k=0}^n k^p y^k (1+y)^{n-k} = (1+y)^{n+1} w_p(y) - y^{n+1} \sum_{k=0}^p \binom{p}{k} (n+1)^{p-k} w_k(y).$$

Integrating both sides of the above equation with respect y from -1 to 0 , we have

$$\begin{aligned}
& \sum_{k=0}^n k^p \int_{-1}^0 y^k (1+y)^{n-k} dy \\
&= \int_{-1}^0 (1+y)^{n+1} w_p(y) dy - (n+1)^p \int_{-1}^0 y^{n+1} dy - p(n+1)^{p-1} \int_{-1}^0 y^{n+2} dy \\
&\quad - \sum_{k=2}^p \binom{p}{k} (n+1)^{p-k} \int_{-1}^0 y^{n+1} w_k(y) dy. \tag{33}
\end{aligned}$$

Using (30) in the left hand side of (33) yields

$$\sum_{k=0}^n k^p \int_{-1}^0 y^k (1+y)^{n-k} dy = \frac{1}{n+1} \sum_{k=0}^n \frac{(-1)^k}{\binom{n}{k}}.$$

From (8), the first integral in the right hand side of (33) becomes

$$\int_{-1}^0 (1+y)^{n+1} w_p(y) dy = \frac{1}{n+2} B_{p,n+1}.$$

The second and third integrals in the right hand side can be evaluated easily. For the last integral in right hand side of (33), if we use (26) and (22), we have

$$\begin{aligned}
& \sum_{k=2}^p \binom{p}{k} (n+1)^{p-k} \int_{-1}^0 y^{n+1} w_k(y) dy \\
&= \frac{(-1)^{n+p} (n+1)^p}{n+2} + \frac{(-1)^{n+1} p (n+1)^{p-1}}{n+3} - \frac{(-1)^{n+p} B_{p,n+1} (-n)}{n+2}.
\end{aligned}$$

Finally, combining all these evaluated integrals give the desired equation. ■

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